JOURNAL OF APPROXIMATION THEORY 7, 391–394 (1973)

On Weak Underpolynomials of Generalized Infrapolynomials

BRUCE L. CHALMERS

Department of Mathematics, University of California, Riverside, California 92502

Communicated by Joseph L. Walsh

Received October 18, 1970

Let P_n denote the class of polynomials $\sum_{i=0}^{n} c_i z^i$ with complex coefficients considered as mappings of the complex z-plane $\mathbf{C}_z$ into itself. Let $L = \{\mathscr{L}^i\}_{i=0}^{r}$ denote a fixed set of $r + 1$ linearly independent linear functionals on P_n, and let $A = (A_0, A_1, ..., A_r)$ be a fixed $(r + 1)$-tuple of complex numbers. Then $P_n(A)$ will represent the class of polynomials $p(z)$ in P_n such that $\mathscr{L}^i p = A_i$, $i = 0, 1, ..., r$. Further, let E denote a compact subset of $\mathbf{C}_z$ containing at least $n - r$ points. Following the work of many authors (see, e.g., [3]), we make the following

DEFINITION. $p(z) \in P_n(A)$ is called an infropolynomial on E with respect to $P_n(A)$ if $p(z)$ has on E no underpolynomials with respect to $P_n(A)$; i.e., if there exists no polynomial $q(z)$ in $P_n(A)$ such that

$$|q(z)| < |p(z)| \quad \text{on} \quad E \cap \{z; p(z) \neq 0\}, \tag{1}$$

$$q(z) = 0 \quad \text{on} \quad E \cap \{z; p(z) = 0\}. \tag{2}$$

A polynomial $q(z) \in P_n(A)$ such that $q(z) \not\equiv p(z)$ and

$$|q(z)| \leqslant |p(z)| \text{ on } E$$

is called a weak underpolynomial of $p(z)$ on E with respect to $P_n(A)$.

In [3], Zedek obtained the following theorem, extending a result due to Motzkin and Walsh [2], who proved the case $r = 0$.

THEOREM 1. *Suppose $\mathscr{L}^i(p) = p^{(n-i)}(0) = (n - i)!\, c_{n-i}$, $i = 0, 1, ..., r$. If $p(z) \in P_n(A)$ is an infrapolynomial on E with respect to $P_n(A)$, then $p(z)$ has no weak underpolynomial on E with respect to $P_n(A)$.*

DEFINITION. Let e_z^i denote the linear functional defined on P_n by $e_z^i p = p^{(i)}(z)$. For notational convenience in the sequel, the $(m + 1)$-tuple

$\{e^0_{z_\beta}, e^1_{z_\beta}, \ldots, e^m_{z_\beta}\}$ will be denoted by $\{e^0_{z_\beta}, e^0_{z_\beta}, \ldots, e^0_{z_\beta}\}$, and in any k-tuple of points $(z_1, z_2, \ldots, z_k)$ it will be assumed that if $z_i = z_j$ then $z_{i+k} = z_i$, $0 \leqslant k \leqslant j - i$. If $\{\mathscr{L}^0, \mathscr{L}^1, \ldots, \mathscr{L}^r, e^0_{z_1}, \ldots, e^0_{z_{n-r}}\}$ forms a linearly independent set in the dual of P_n for each $(n - r)$-tuple of points $z_1, z_2, \ldots, z_{n-r}$ in E, then we will say that E is nonsingular for (L, P_n). Otherwise we will say E is singular for (L, P_n).

We will prove the following theorem.

THEOREM 2. *If E is nonsingular for (L, P_n), and $p(z) \in P_n(A)$ is an infra-polynomial on E with respect to $P_n(A)$, then $p(z)$ has no weak underpolynomials on E with respect to $P_n(A)$. As a partial converse we have that if E is singular for (L, P_n) and contains precisely $n - r$ points, then there exists an infra-polynomial $p(z)$ with a weak underpolynomial on E with respect to $P_n(A)$.*

EXAMPLE 1. Theorem 1 is a special case of Theorem 2 since, in Theorem 1, E is nonsingular for (L, P_n). Indeed, if $p(z) \in P_n$ and $\mathscr{L}^i(p) = p^{(n-i)}(0) = (n - i)!\, c_{n-i} = 0 (i = 0, 1, \ldots, r)$, then $p(z) \in P_{n-r-1}$. Since $\{e^0_{z_1}, \ldots, e^0_{z_{n-r}}\}$ is an Hermite system on P_{n-r-1}, we see that $e^0_{z_1}(p) = 0, \ldots, e^0_{z_{n-r}}(p) = 0$ implies $p(z) = 0$. Thus $\{\mathscr{L}^0, \ldots, \mathscr{L}^r, e^0_{z_1}, \ldots, e^0_{z_{n-r}}\}$ is a linearly independent set in the dual of P_n for any points $z_1, z_2, \ldots, z_{n-r}$ in E.

It follows from a result due to D. R. Ferguson [1, p. 20] that if $\mathscr{L}^i = e_0^{j_i}$ $(i = 0, 1, \ldots, r)$, the choice $j_i = n - i$ $(i = 0, 1, \ldots, r)$, as in the case of Theorem 1, is the only configuration such that every E is nonsingular.

EXAMPLE 2. Suppose (i) $\mathscr{L}^i(p) = p^{(n-i)}(0) = c_{n-i}$, $i = 0, 1, \ldots, k - 1$, (ii) $\mathscr{L}^i(p) = p^{(i-k)}(0) = c_{i-k}$, $i = k, k + 1, \ldots, r$, and (iii) $0 \notin E$. Then E is nonsingular for (L, P_n). This is clear, since $c_{n-i} = 0$ $(i = 0, 1, \ldots, k - 1)$ and $c_{i-k} = 0$ $(i = k, k + 1, \ldots, r)$ imply $p(z) \in z^{r-k+1}P_{n-r-1}$, and, thus, if $e^0_{z_1}(p) = 0, \ldots, e^0_{z_{n-r}}(p) = 0$, where no $z_\alpha = 0$, then $p(z) \equiv 0$. ($e^0_{z_1}, \ldots, e^0_{z_{n-r}}$ is an Hermite system on the space $z^{r-k+1}P_{n-r-1}$ which is a Haar space on E, since $0 \notin E$.)

It is also a consequence of Ferguson's result mentioned above that if $\mathscr{L}^i = e_0^{j_i}$ $(i = 0, 1, \ldots, r)$, and E is nonsingular for (L, P_n) whenever $0 \notin E$, then the j_i must be as in Example 2.

Suppose $\mathscr{L}^i = e_{w_i}^{j_i}$ $(i = 0, 1, \ldots, r)$ and the set $N_k = \{j_i\,; j_i \geqslant n - k\}$ contains no more than $k + 1$ elements, $k = 0, 1, \ldots, r$. Then from another result of Ferguson [1, pp. 4, 8], we have that the set of $(n - r)$-tuples $(z_1, z_2, \ldots, z_{n-r})$ such that $\{\mathscr{L}^0, \ldots, \mathscr{L}^r, e^0_{z_1}, \ldots, e^0_{z_{n-r}}\}$ is a linearly dependent set in the dual of P_n, is a closed, nowhere dense subset of the complex $(n - r)$-space.

EXAMPLE 3. Let $n = 3$, $r = 1$, $\mathscr{L}^0 = e_0{}^3$, $\mathscr{L}^1 = e_0{}^1$, $A_0 = 3!$, and

$A_1 = 0$. Then $P_3(A) = \{z^3 + az^2 + b\}$. First, let $E = \{-k, k\}$ for some $k > 0$. Then E is singular for (L, P_3), for, if $p(z) \in P_3$ and $\mathscr{L}^0 p = \mathscr{L}^1 p = 0$, then $p(z) = cz^2 + d$ which vanishes throughout E if $d = -ck^2$. Hence, according to the second part of Theorem 2, there exists an infrapolynomial with a weak underpolynomial on E with respect to $P_3(A)$.

Secondly, let $E = \{-k, m\}$, where $m > 0$, $k > 0$, $m \neq k$. Then, by the above argument, E is nonsingular for (L, P_3).

EXAMPLE 4. Let $n = 3$, $r = 1$, $\mathscr{L}^0 = 2e_0{}^3 - e_0{}^1$, $\mathscr{L}^1 = e_0{}^0$, $A_0 = 1$, $A_1 = 2$, and $E = \{1, 2, 3\}$. Then $P_3(A) = az^3 + bz^2 + (12a - 1)z + 2$. Further, E is nonsingular for (L, P_3). For, if $\mathscr{L}^0 p = \mathscr{L}^1 p = 0$, then $p(z) = az^3 + bz^2 + 12az$. If a were $\neq 0$, the product of the nonzero zeros of $p(z)$ would be 12. But no two points of E have 12 as their product.

We now adapt the method of [2] and [3] to obtain

Proof of Theorem 2. Suppose that $p(z) \in P_n(A)$ has on E a weak underpolynomial $r(z)$ with respect to $P_n(A)$, where E is nonsingular for (L, P_n). We will demonstrate the existence on E of an underpolynomial $q(z)$ with respect to $P_n(A)$. Let $m(z) = \frac{1}{2}[r(z) + p(z)]$. Then $m(z) \in P_n(A)$, and, clearly, for each z in E, either $|m(z)| < |p(z)|$ or $m(z) = p(z)$. Let $m(z) = m_1(z) f(z)$, and $p(z) = p_1(z) f(z)$, where $f(z)$ is the k-th degree monic polynomial whose zeros $z_1, z_2, \ldots, z_k$ are precisely the common zeros of $m(z)$ and $p(z)$ in E, multiple zeros being repeated.

Consider $m(z) - p(z)$. We have $\mathscr{L}^i(m - p) = 0$ $(i = 0, 1, \ldots, r)$, and, further, $e_{z_1}^0(m - p) = 0, \ldots, e_{z_k}^0(m - p) = 0$. Hence, if $S = \{w_1, \ldots, w_t\}$ is the subset of E on which $m(z) = p(z) \neq 0$, and, thus, where $m_1(z) = p_1(z) \neq 0$, then $0 \leqslant t \leqslant n - r - k - 1$, by our hypothesis that E is nonsingular for (L, P_n).

From this hypothesis we also obtain the existence of an $L^*(z) \in P_n$ such that $\mathscr{L}^i(L^*) = 0$ $(i = 0, 1, \ldots, r)$, $e_{z_1}^0(L^*) = 0, \ldots, e_{z_k}^0(L^*) = 0$, and $L^*(z) = p_1(z) f(z) = m_1(z) f(z)$ for all $z \in S$. Let $L(z) = L^*(z)/f(z)$.

Since $|m_1(z) - L(z)| = 0 < |p_1(z)|$ for $z \in S$, the same inequality holds for some open neighborhood U of S. Hence for all e, $0 < e < 1$, $|m_1(z) - eL(z)| = |e[m_1(z) - L(z)] + (1 - e) m_1(z)| < |p_1(z)|$ for $z \in U$.

Next, since $|m_1(z)| < |p_1(z)|$ for $z \in E - S$, we have that $|m_1(z)| < |p_1(z)|$ on the compact set $E - U$. Thus, for e sufficiently small,

$$|m_1(z) - eL(z)| < |p_1(z)| \qquad \text{for} \quad z \in E - U.$$

We conclude that $|m_1(z) - eL(z)| < |p_1(z)|$ on E if e is sufficiently small. Hence $|m(z) - eL^*(z)| < |p(z)|$ on $E \cap \sim\{z_1, z_2, \ldots, z_k\} = E \cap \{z; p(z) \neq 0\}$ for e sufficiently small, and $m(z) - eL^*(z) = 0$ on $E \cap \{z; p(z) = 0\}$. We must still check that $m(z) - eL^*(z) \in P_n(A)$, but this holds since $\mathscr{L}^i(L^*) = 0$ $(i = 0, 1, \ldots, r)$.

For the partial converse, assume that $p(z)$ is an element of $P_n(A)$ of smallest supremum norm on E. Then $p(z)$ is an infrapolynomial on E with respect to (L, P_n). Moreover, since E is singular for (L, P_n), and contains precisely $n - r$ points, $\{z_\alpha\}_{\alpha=1}^{n-r}$, there exists an $\epsilon(z) \in P_n$, $\epsilon(z) \not\equiv 0$, such that $\mathscr{L}^i(\epsilon) = 0$, $i = 0, 1, \ldots, r$, and $\epsilon(z_\alpha) = 0$ $(\alpha = 1, 2, \ldots, n - r)$. Hence, $p(z) + \epsilon(z)$ is a weak underpolynomial of $p(z)$ on E with respect to $P_n(A)$. $\square$

References

1. D. R. Ferguson, The question of uniqueness for G. D. Birkhoff interpolation problems, *J. Approximation Theory* **2** (1969), 1–28.
2. T. S. Motzkin and J. L. Walsh, Underpolynomials and infrapolynomials, *Illinois J. Math.* **1** (1957), 406–426.
3. M. Zedek, On the definition of restricted infrapolynomials, *J. Approximation Theory* **1** (1968), 251–254.